Line Integrals

Alex Alemi

January 27, 2010

In physics, and in particular in this class we will deal a lot with scalar and vector fields. These fields are all good and well, but as physicists we will often need to answer particular questions about these fields. To answer such questions we employ the power of vector calculus.

One particular quantity of interest which we will see again and again in our studies is how much of our field lies along a certain path or contour (a contour is a closed path).

1 What it is

Consider a scalar field $T(\boldsymbol{x}) = T(x, y, z)$ (imagine the temperatures in a room) defined in Cartesian coordinates and a vector field $\boldsymbol{v}(x, y, z)$ (imagine velocities of the air molecules in the room).

Now imagine some path which is of interest (for concreteness imagine the hem of your coat), the contour C. This could be any squiggle you could imagine drawn in the room, hanging in the air stationary. What we aim to calculate is how 'much' of our field lies along this contour.

1.1 Scalar Field

In the case of the scalar field, the answer suggests itself. In order to construct the line integral of T, i.e.

$$\int_C T(\boldsymbol{x}) ds$$

we can imagine the procedure as a Riemann sum. Cut up our contour in a series of chunks p_i. Evaluate our function on each chunk, multiply by the size of the chunk and sum all of these together. But at which point in our chunk do we evaluate our function? Here comes the beauty of continuous functions. If our scalar field is sufficiently nice (i.e. continuous), it doesn't matter in the least which point we choose to evaluate our function since we will take as the integral the limit of cutting our contour into vanishing chunks.

So, consider some set of points inside of each chunk $\boldsymbol{x}_i$. Now our first approximation suggests itself. The size of the ith chunk can also be approximated as $\Delta x_i = |\boldsymbol{x}_i - \boldsymbol{x}_{i+1}|$.

$$\int_C T(\boldsymbol{x}) ds \approx \sum_i T(\boldsymbol{x}_i) \Delta x_i$$

The true answer of course will be the limit of this sum as we take vanishing chunks.

1.2 Vector Field

Fine and good, but what is to be done with our vector field? We must decide what we mean but how 'much' of the vector lies along the contour. At this point we are interested in computing some number in the end, so it will not do to simply replace $T(\boldsymbol{x})$ with $\boldsymbol{v}(\boldsymbol{x})$ above. This would produce a legitimate mathematical quantity, but require us to perform a rather tedious and complicated vector sum. Instead we compute

$$\int_C \boldsymbol{F}(\boldsymbol{x}) \cdot d\boldsymbol{l}$$

Instead we will again chunk up our path, but recognize that our path contains more information than simply its positions, the fact that it is a path means that at any point we also have some notion of which direction we are headed.

So now at each point in our path, imagine not a simple chunk, but section up our path into little arrows. I.e. at a discrete set of points along our contour we construct a vector, whose direction indicates the way the curve is headed, i.e. it lies tangent to the curve. But what of its length? We must of course preserve the additional notion of the length of our path if we are to make any sense in our integration.

So the procedure is as follows. Take our contour. Chunk it up. Pick points $\boldsymbol{x}_i$. And then for each chunk create a vector whose direction indicates the way the curve is headed, i.e. lie it tangent to the curve, and make its length the length of the chunk in question. We obtain the set of vectors I will denote $\boldsymbol{l}_i$.

Now in order to figure out how much of our vector field lies along our contour, we will utilize the dot product (which does exactly that).

We have the suggested approximation

$$\int_C \boldsymbol{F}(\boldsymbol{x}) \cdot d\boldsymbol{l} \approx \sum_i \boldsymbol{F}(\boldsymbol{x}_i) \cdot \boldsymbol{l}_i$$

Where again our actual answer will be the result obtained in the limit that this procedure is iterated in the limit of vanishing chunks.

2 How it's done

Its all fine and dandy to talk through the procedure we will take, but that final step remains troublesome. How exactly are we supposed to repeat this procedure in the limit of vanishing chunks, surely I can't seriously be asking you to perform this whole procedure, and then repeat it all over again for a smaller set of chunks. And then to ask that you do this an infinite number of times?

Here is where Newton's calculus saves the day. In practice, with careful manipulations of d's, you can arrive at the exact solution with little work.

The way the problem will work is that you will often be presented not with some squiggle on a page, but with a set of equations parameterizing your path. (Of course these could be constructed for a squiggle, but that would require a lot of work with little rulers and compasses). Given a parametrization of your path, i.e. a set of equations, we can simply take derivatives and arrive at our answer.

What do I mean by parametrization. I mean that some kind soul has walked the path ahead of you and placed markers of some kind. For concreteness, consider the mile markers on a curvy highway. I can parametrize the highway by constructing the function $\boldsymbol{LL}(mile)$, which when given an odometer reading (in miles), returns the latitude and longitude you'll have at that point in your journey. This function parametrizes the highway according the the highways arc length. I could also construct a parametrization $\boldsymbol{LL}(fuel)$ which returns your latitude and longitude for a particular reading of your fuel gauge (which you should convince yourself is not the same as $\boldsymbol{LL}(mile)$,[think hills]). This would be a much more complicated parametrization, but a parametrization none-the-less.

So now I could describe the line integral in the most general case. Consider a curve C, with a parametrization $\boldsymbol{r}(s)$. We want to perform our sum by ticking off the parametrization parameter s. But we need to be able to construct our $\boldsymbol{dl}$ vector. We need to figure out what vector is tangent to our curve and has a length associated with motion along s.

Lets talk it out. When I am at tick s, I'll be at location $\boldsymbol{r}(s)$. If I move a tiny tick Δs, I'll be at the location $\boldsymbol{r}(s + \Delta s)$. What vector describes not only which direction I moved but how far I moved? Hopefully you can convince yourself that it would be

$$\Delta \boldsymbol{r}(s) = \boldsymbol{r}(s + \Delta s) - \boldsymbol{r}(s)$$

but of course we want to do calculus, recall the definition of the derivative

$$\boldsymbol{r}'(s) = \frac{d\boldsymbol{r}(s)}{ds} = \lim_{\Delta s \to 0} \frac{\boldsymbol{r}(s + \Delta s) - \boldsymbol{r}(s)}{\Delta s}$$

This is precisely what we want to do, take the limit as $\Delta s \to 0$, i.e. infinitesimal chunks. So we can write

$$\lim_{\Delta s \to 0} \Delta \boldsymbol{r}(s) = \frac{d\boldsymbol{r}(s)}{ds} ds = \boldsymbol{r}'(s)\ ds$$

And we have obtained in the general case, for a vector field $\boldsymbol{v}(\boldsymbol{x})$, contour C, with starting tick s_i and final tick s_f, with parametrization $\boldsymbol{r}(s)$

$$\int_C \boldsymbol{v} \cdot d\boldsymbol{l} = \int_{s_i}^{s_f} \boldsymbol{v}(\boldsymbol{r}(s)) \cdot \boldsymbol{r}'(s)\ ds$$

This formula looks simple, but please take a minute to realize what exactly has transpired from the left hand side to the right. Notice also that while the left hand side is rather cryptic, the right hand side offers a real procedure for calculating the line integral, you need only take your vector function $\boldsymbol{r}(s)$, take derivatives of that function (ordinary one dimensional derivatives you know and love), take a dot product (which you know how to do), and then you'll have a one dimensional integral in a single variable (the integrals you know and love how to do). I.e. the right hand side actually tells you how to do a line integral.

I took a rather round about way to get here, but hopefully I made the simple formula (which you can always look up in the future) understandable.

3 Example

Alright, lets get our feet dirty and perform an actual computation. We will use the result obtained above

$$\int_C \boldsymbol{v} \cdot d\boldsymbol{l} = \int_{s_i}^{s_f} \boldsymbol{v}(\boldsymbol{r}(s)) \cdot \boldsymbol{r}'(s)\ ds$$

Consider the vector field

$$\boldsymbol{F}(\boldsymbol{x}) = xy\hat{\imath} + \sin\frac{y\pi}{2}\hat{\jmath} + xz^3\hat{k} = \left(xy, \sin\frac{y\pi}{2}, xz^3\right)$$

and lets consider the contour defined by $z = \sin\frac{x\pi}{2}$, $y = x^2$, from the point $(0,0,0)$ to the point $(1,1,1)$. Lets make our life simple and parametrize by x, (so we integrate from $x = 0$ to $x = 1$) obtaining the parametrization

$$\boldsymbol{r}(x) = \left(x, x^2, \sin\frac{x\pi}{2}\right)$$

from which we have

$$\boldsymbol{r}'(x) = \frac{d\boldsymbol{r}(x)}{dx} = \left(1, 2x, -\frac{\pi}{2}\cos\frac{x\pi}{2}\right)$$

At which point its straight forward to compute

$$
\begin{aligned}
\int_C \boldsymbol{F} \cdot d\boldsymbol{l} &= \int \boldsymbol{F}(\boldsymbol{r}(x)) \cdot \boldsymbol{r}'(x)\ dx \\
&= \int_0^1 dx\ \left(x^2, \sin\frac{x\pi}{2}, x\sin^3\frac{x\pi}{2}\right) \cdot \left(1, 2x, -\frac{\pi}{2}\cos\frac{x\pi}{2}\right) \\
&= \int_0^1 dx\ \left[x^2 + 2x\sin\frac{x\pi}{2} - \frac{x\pi}{2}\sin^3\frac{x\pi}{2}\cos\frac{x\pi}{2}\right] \\
&= \frac{1}{3} + \int_0^1 dx\ 2x\sin\frac{x\pi}{2} - \int_0^1 dx\ \frac{x\pi}{2}\sin^3\frac{x\pi}{2}\cos\frac{x\pi}{2} \\
&= \frac{1}{3} + \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}} d\xi\ \xi\sin\xi - \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\xi\ \xi\sin^3\xi\cos\xi \\
&= \frac{1}{3} + \frac{8}{\pi^2} - \frac{2}{\pi}\frac{5\pi}{64} \\
&= \frac{17}{96} + \frac{8}{\pi^2} \approx 0.99
\end{aligned}
$$

Tada.